International Journal of Theoretical Physics, Vol. 29, No. 8, 1990

Static Spherically Symmetric Solutions in General Projective Relativity

T. Singh[1] and G. P. Singh[1]

Received October 25, 1989

Static spherically symmetric solutions have been obtained for general projective relativity with $n=0$ and $n\neq 0$ both in isotropic and curvature coordinates. In curvature coordinates, only a restricted exact solution is possible. However, an approximate solution can always be obtained following a method similar to Vanden Bergh. In these spacetimes there is no horizon, but only a naked singularity at $r=0$. Thus there are no black holes. It is shown that there is no solution in static, spherically symmetric, conformally flat spacetime.

1. INTRODUCTION

Recently Arcidiacono (1986) developed a new general projective relativity (GPR) which is based on the de Sitter universe. The local curvature is described by the generalized Einstein equations

$$R_{AB}-\tfrac{1}{2}R\gamma_{AB}=\chi T_{AB} \qquad (A, B=0,1,2,3,4) \tag{1.1}$$

where γ_{AB} is the five-dimensional metric and T_{AB} is the energy tensor of the material field. These equations are similar to the equations of Jordon-Thiry unified theory (which generalizes the Kaluza-Klein five-dimensional theory), but have a different physical interpretation.

As a particular case, Arcidiacono (1987) obtained the equations of the scalar-tensor gravitational field, which have only a formal similarity with the Brans-Dicke field (Brans and Dicke, 1961). The new field equations are

$$\begin{aligned}\hat{R}_{ik}-\tfrac{1}{2}a_{ik}\hat{R}+(3n+1)\phi^{-1}[\nabla_i\nabla_k\phi-a_{ik}\Box\phi\\ -3n\phi^{-2}[(n+1)\nabla_i\phi\nabla_k\phi+na_{ik}\nabla_l\phi\nabla_s\phi a^{ls}]=\chi\phi^{-2}T_{ik}\end{aligned} \tag{1.2}$$

$$\hat{R}+6n[\phi^{-1}\Box\phi+(n-1)\phi^{-2}\nabla_l\phi\nabla_s\phi a^{ls}]=-2\chi\phi^{-4}T_{00} \tag{1.3}$$

where i, $k=1$, 2, 3, 4 and the quantities with carets refer to their four-dimensional components.

[1]Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221005, India.

0020-7748/90/0800-0873$06.00/0

For the vacuum case, the field equations of general projective relativity with $n=0$ are

$$\hat{R}_{ik}+\frac{1}{\phi}\phi_{;ik}=0 \tag{1.4}$$

$$\Box\phi\equiv a^{ik}\phi_{;ik}=0 \tag{1.5}$$

Here a semicolon denotes covariant derivatives in four-dimensional spacetime.

Arcidiacono (1986) has given a technique to obtain a solution of the GPR field equations for arbitrary n from a solution of the GPR field equations with $n=0$ by the transformation of the metric as

$$\gamma_{ik}=\phi^{2n}a_{ik}; \qquad \gamma_{00}=\phi^{2(n+1)}; \qquad \gamma_{i0}=0 \tag{1.6}$$

Singh and Arcidiacono (1989) have established an analogue of Birkhoff's theorem in GPR, and Singh and Singh (1989) have considered a product space and plane-fronted waves in GPR. Singh *et al.* (1989) have considered a stationary axisymmetric vacuum field in GPR.

In this paper we consider the vacuum fields of GPR in spherically symmetric spacetime and obtain solutions of GPR with $n=0$ and $n\neq 0$. Finally, the nature of singularities of these solutions is discussed.

2. SPHERICALLY SYMMETRIC SOLUTIONS IN ISOTROPIC COORDINATES

We consider the static, spherically symmetric spacetime whose metric is in the isotropic form (Synge, 1960)

$$ds^2=e^{\alpha}\,dt^2-e^{\beta}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2) \tag{2.1}$$

where α and β are functions of r alone.

Taking ϕ as a function of r only, we find that the field equations (1.4) and (1.5) for the metric (2.1) reduce to

$$\beta_{11}+\frac{\alpha_{11}}{2}+\frac{\alpha_1^2}{2}+\frac{\beta_1}{r}-\frac{\alpha_2\beta_1}{4}+\frac{\phi_{11}}{\phi}-\frac{\beta_1\phi_1}{2\phi}=0 \tag{2.2}$$

$$\beta_{11}+\frac{\beta_1^2}{2}+\frac{3\beta_1}{r}+\frac{\alpha_1\beta_1}{2}+\frac{\alpha_1}{r}+\frac{\beta_1\phi_1}{\phi}+\frac{2\phi_1}{r\phi}=0 \tag{2.3}$$

$$\alpha_{11}+\frac{\alpha_1^2}{2}+\frac{\alpha_2\beta_1}{2}+\frac{2\alpha_1}{r}+\frac{\alpha_1\phi_1}{\phi}=0 \tag{2.4}$$

$$\phi_{11}+\frac{(\alpha_1+\beta_1)\phi_1}{2}+\frac{2\phi_1}{r}=0 \tag{2.5}$$

The subscript 1 denotes differentiation with respect to r.

From equations (2.3) and (2.4), we have

$$\frac{\alpha_{11}+\beta_{11}}{2}+\frac{(\alpha_1+\beta_1)\phi_1}{2\phi}+\frac{(\alpha_1+\beta_1)}{2}\left(\frac{\alpha_1+\beta_1}{2}+\frac{3}{r}\right)+\frac{\phi_1}{r\phi}=0 \tag{2.6}$$

From equations (2.5) and (2.6), we obtain

$$\frac{\alpha_{11}+\beta_{11}}{2}+\frac{(\alpha_1+\beta_1)\phi_1}{2\phi}+\frac{\phi_{11}}{\phi}+\left(\frac{\alpha_1+\beta_1}{2}+\frac{\phi_1}{\phi}\right)\left(\frac{\alpha_1+\beta_1}{2}+\frac{3}{r}\right)=0 \tag{2.7}$$

which on integration gives

$$r^2\phi\, e^{(\alpha+\beta)/2}=ar^2+b \tag{2.8}$$

where a and b are constants.

From equations (2.4) and (2.5), we have

$$\phi\alpha_{11}+\alpha_1\phi_1+\phi_{11}+(\alpha_1\phi+\phi_1)\left(\frac{\alpha_1+\beta_1}{2}+\frac{2}{r}\right)=0 \tag{2.9}$$

After integration, this gives

$$r^2\, e^{(\alpha+\beta)/2}(\alpha_1\phi+\phi_1)=p \tag{2.10}$$

where p is a constant of integration.

We can write equation (2.5) as

$$\frac{d}{dr}\left(r\, e^{(\alpha+\beta)/2}\phi_1\right)=0 \tag{2.11}$$

On integration, equation (2.11) gives

$$r^2\, e^{(\alpha+\beta)/2}\phi_1=K \tag{2.12}$$

Here K is a constant.

Dividing equation (2.12) by (2.8) and further integrating, we obtain

$$\phi=m_1\exp\left\{\frac{K}{(ab)^{1/2}}\tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2}r\right]\right\} \tag{2.13}$$

Subtracting equation (2.12) from (2.10) and then dividing by (2.8), we get

$$\alpha_1=\frac{p-K}{ar^2+b} \tag{2.14}$$

which has the solution

$$e^{\alpha}=m_2\exp\left\{\frac{p-K}{(ab)^{1/2}}\tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2}r\right]\right\} \tag{2.15}$$

Here m_1 and m_2 are constants of integration. Substituting the values of ϕ and e^α from equations (2.13) and (2.15) in (2.8), we have

$$e^\beta = \frac{(ar^2+b)^2}{m_1^2 m_2 r^4} \exp\left\{\frac{-(K+p)}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \tag{2.16}$$

In order for these solutions to satisfy all the field equations, we must have a relation between constants such as

$$p^2 + 3K^2 + 16ab = 0 \tag{2.17}$$

Thus equations (2.13)-(2.17) constitute the complete solution in GPR (with $n=0$) for the metric (2.1). The corresponding metric is given by

$$\begin{aligned} ds^2 = {} & m_2 \exp\left\{\frac{p-K}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} dt^2 \\ & - \frac{(ar^2+b)^2}{m_1^2 m_2 r^4} \exp\left\{\frac{-(K+p)}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ & \times (dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \end{aligned} \tag{2.18}$$

The GPR vacuum solution for arbitrary n via transformation (1.6) are

$$\begin{aligned} \phi &= m_1 \exp\left\{\frac{K}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{11} &= \frac{(ar^2+b)^2}{m_1^{2(1-n)} m_2 r^4} \exp\left\{\frac{2nK-K-p}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{22} &= -\frac{(ar^2+b)^2}{m_1^{2(1-n)} m_2 r^2} \exp\left\{\frac{2nK-K-p}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{33} &= -\frac{(ar^2+b)^2 \sin^2\theta}{m_1^{2(1-n)} m_2 r^2} \exp\left\{\frac{2nK-K-P}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{44} &= m_1^{2n} m_2 \exp\left\{\frac{2nK-K+p}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{00} &= m_1^{2(n+1)} \exp\left\{\frac{2K(n+1)}{(ab)^{1/2}} \tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \\ \gamma_{10} &= 0 \end{aligned} \tag{2.19}$$

There is no horizon in this spacetime, only a naked singularity at $r=0$. Thus, there are no black holes.

3. SPHERICALLY SYMMETRIC SOLUTIONS IN CURVATURE COORDINATES

We take the spherically symmetric line element in the form (Synge, 1960)

$$ds^2 = e^\alpha \, dt^2 - e^\beta \, dr^2 - r^2 \, d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{3.1}$$

where α and β are functions of the radial coordinate r alone.

Considering the scalar field ϕ as a function of r only, the field equations (1.4) and (1.5) for the line element (3.1) reduce to

$$\frac{\alpha_{11}}{2} + \frac{\alpha_1(\alpha_1 - \beta_1)}{4} - \frac{\beta_1}{r} + \frac{\phi_{11}}{\phi} - \frac{\beta_1\phi_1}{2\phi} = 0 \tag{3.2}$$

$$1 + \frac{r(\alpha_1 - \beta_1)}{2} + \frac{r\phi_1}{\phi} - e^\beta = 0 \tag{3.3}$$

$$\frac{\alpha_{11}}{2} + \frac{\alpha_1(\alpha_1 - \beta_1)}{4} + \frac{\alpha_1}{r} + \frac{\alpha_1\phi_1}{2\phi} = 0 \tag{3.4}$$

$$\phi_{11} + \frac{(\alpha_1 - \beta_1)\phi_1}{2} + \frac{2\phi_1}{r} = 0 \tag{3.5}$$

After a recombination and rearrangement of terms, equations (3.2)-(3.5) reduce to the following equations:

$$-r\beta_1 - \frac{r^2\beta_1\phi_1}{2\phi} + \frac{r^2\phi_{11}}{\phi} + \frac{2r\phi_1}{\phi} + 1 - e^\beta = 0 \tag{3.6}$$

$$\frac{r\alpha_{11}}{\alpha_1} + 1 + e^\beta = 0 \tag{3.7}$$

$$r\beta_1 + \frac{r^2\alpha_1\phi_1}{2\phi} - 1 + e^\beta = 0 \tag{3.8}$$

$$r\left(\frac{\phi_{11}}{\phi_1} - \frac{\phi_1}{\phi}\right) + 1 + e^\beta = 0 \tag{3.9}$$

From equations (3.7) and (3.9), we get

$$\alpha_1 = m\frac{\phi_1}{\phi} \tag{3.10}$$

which gives the solution as

$$e^\alpha = p\phi^m \tag{3.11}$$

where m and p are constants.

Using (3.10) in (3.8), we have

$$r\beta_1+\frac{m}{2}\left(\frac{r\phi_1}{\phi}\right)^2-1+e^{\beta}=0 \tag{3.12}$$

From equations (3.6), (3.9), and (3.12), we obtain

$$\frac{r^2\phi_{11}}{2\phi}+\frac{m}{4}\left(\frac{r\phi_1}{\phi}\right)^3+\frac{m+1}{2}\left(\frac{r\phi_1}{\phi}\right)^2+\frac{r\phi_1}{\phi}=0 \tag{3.13}$$

Equations (3.9) and (3.13) then yield

$$e^{\beta}=1+(m+2)\frac{r\phi_1}{\phi}+\frac{m}{2}\left(\frac{r\phi_1}{\phi}\right)^2 \tag{3.14}$$

Integrating (3.5), we get

$$r^2\phi_1=K\,e^{(\beta-\alpha)/2} \tag{3.15}$$

Here K is an integration constant.

Again from (3.11) and (3.15), one gets

$$r^2\left(\frac{r\phi_1}{\phi}\right)^2=K^{-2}\phi^{-(m+2)}\,e^{\beta} \tag{3.16}$$

Now we consider two cases, depending on the value of m.

Case I. When $m=-2$. From equations (3.14) and (3.16), we obtain

$$\frac{\phi_1}{\phi}=(K^2r^4+r^2)^{-1/2} \tag{3.17}$$

which on integration gives

$$\phi=\frac{dKr}{1+(K^2r^2+1)^{1/2}} \tag{3.18}$$

From (3.11) and (3.18), one has

$$e^{\alpha}=\frac{p}{d^2K^2r^2}[1+(K^2r^2+1)^{1/2}]^2 \tag{3.19}$$

Using (3.17) in (3.16), we get

$$e^{\beta}=\frac{K^2r^2}{K^2r^2+1} \tag{3.20}$$

Thus, in this case the line element (3.1) turns into the form

$$ds^2=\frac{p}{d^2K^2r^2}[1+(K^2r^2+1)^{1/2}]^2\,dt^2 - \frac{K^2r^2}{K^2r^2+1}\,dr^2-r^2\,d\theta^2-r^2\sin^2\theta\,d\phi^2 \tag{3.21}$$

Now the GPR vacuum solutions for arbitrary n via the transformation (1.6) are

$$
\begin{aligned}
\phi &= \frac{dKr}{1+(K^2+r^2+1)^{1/2}} \\
\gamma_{11} &= -\frac{K^2r^2}{K^2r^2+1}\left\{\frac{dKr}{1+(K^2r^2+1)^{1/2}}\right\}^{2n} \\
\gamma_{22} &= -r^{2(n+1)}\left\{\frac{dK}{1+(K^2r^2+1)^{1/2}}\right\}^{2n} \\
\gamma_{33} &= -r^{2(n+1)}\sin^2\theta\left\{\frac{dK}{1+(K^2r^2+1)^{1/2}}\right\}^{2n} \\
\gamma_{44} &= p\left\{\frac{dKr}{1+(K^2r^2+1)^{1/2}}\right\}^{2(n-1)} \\
\gamma_{00} &= \left\{\frac{dKr}{1+(K^2r^2+1)^{1/2}}\right\}^{2(n+1)} \\
\gamma_{i0} &= 0
\end{aligned}
\tag{3.22}
$$

There is no horizon in this spacetime, only a naked singularity when $n < 1$. Thus, there are no black holes.

Case II. $m \neq -2$. We define

$$
u = r\frac{\phi_1}{\phi} \tag{3.23}
$$

Using (3.23) into (3.13), we get

$$
\frac{m}{2}u^3 + (m+2)u^2 + u = -ru_1 \tag{3.24}
$$

We can rewrite this equation in the form

$$
\begin{aligned}
-\frac{dr}{r} = \frac{du}{u} - \frac{1}{2}\,\frac{mu+m+2}{mu^2/2+(m+2)u+1}\,du \\
-\frac{1}{2}\,\frac{(m+2)\,du}{mu^2/2+(m+2)u+1}
\end{aligned}
\tag{3.24'}
$$

Due to the last term of (3.24') three cases arise:

Case (i). When $\Delta = (m+2)^2 - 2m > 0$, then equation (2.24) yields

$$
\frac{K_1}{r} = u\left[\frac{mu^2}{2} + (m+2)u + 1\right]^{-1/2}\left[\frac{mu+m+2-\Delta^{1/2}}{mu+m+2+\Delta^{1/2}}\right]^{-(m+2)/2\Delta^{1/2}} \tag{3.25}
$$

Case (*ii*). When $\Delta=(m+2)^2-2m=0$,

$$\frac{K_2}{r}=u\left[\frac{mu^2}{2}+(m+2)u+1\right]^{-1/2}\exp\frac{m+2}{mu+m+2} \tag{3.26}$$

Case (*iii*). When $\Delta=(m+2)^2-2m<0$,

$$\frac{K_3}{r}=u\left[\frac{mu^2}{2}+(m+2)u+1\right]^{-1/2}\exp\left[\frac{m+2}{(-\Delta)^{1/2}}\tan^{-1}\frac{mu+m+2}{(-\Delta)^{1/2}}\right] \tag{3.27}$$

Here K_1, K_2, and K_3 are arbitrary constants.

No exact solution is possible in this case. However, an approximate solution can be obtained for $n=0$ and $n\neq 0$ both following a method similar to Vanden Bergh (1980).

4. NONEXISTENCE OF THE CONFORMALLY FLAT SPHERICALLY SYMMETRIC SOLUTIONS IN GPR

We consider the static, spherically symmetric, conformally flat metric in the form

$$ds^2=e^{\alpha}(dt^2-dr^2-r^2\,d\theta^2-r^2\sin^2\theta\,d\phi^2) \tag{4.1}$$

where α is a function of radial coordinate r only.

Assuming ϕ as a function of r, we have for the field equations (1.4) and (1.5) for the metric (4.1)

$$\tfrac{3}{2}\alpha_{11}+\frac{\alpha_1}{r}+\frac{\phi_{11}}{\phi}-\frac{\alpha_1\phi_1}{2\phi}=0 \tag{4.2}$$

$$\tfrac{1}{2}\alpha_{11}+\tfrac{1}{2}\alpha_1^2+\frac{2\alpha_1}{r}+\frac{\alpha_1\phi_1}{2\phi}+\frac{\phi_1}{r\phi}=0 \tag{4.3}$$

$$\tfrac{1}{2}\alpha_{11}+\tfrac{1}{2}\alpha_1^2+\frac{\alpha_1}{r}+\frac{\alpha_1\phi_1}{2\phi}=0 \tag{4.4}$$

$$\phi_{11}+\phi_1\alpha_1+\frac{2}{r}\,\phi_1=0 \tag{4.5}$$

From (4.3) and (4.4), we have

$$\alpha_1+\frac{\phi_1}{\phi}=0 \tag{4.6}$$

which yields the solution

$$\phi=\phi_0\,e^{-\alpha} \tag{4.7}$$

where ϕ_0 is an integration constant.

Using equation (4.6) in (4.4), we get

$$\frac{\alpha_{11}}{\alpha_1}+\frac{2}{r}=0 \tag{4.8}$$

which leads to

$$\alpha=-\frac{K_1}{r}+K_2 \tag{4.9}$$

Here K_1, K_2 are constants of integration.

With the help of equation (4.9), equation (4.7) can be written as

$$\phi=\phi_0\, e^{K_1/r-K_2} \tag{4.10}$$

Now, we see that the equations (4.9) and (4.10) can satisfy the field equation (4.2) if and only if $K_1=0$, which implies that $\alpha=$ const and the scalar field ϕ is constant everywhere. Thus, a conformally flat static solution does not exist in the spherically symmetric case.

5. SINGULAR BEHAVIOR OF THE RIEMANN CURVATURE INVARIANT *R* AND THE KRETSCHMANN CURVATURE INVARIANT *S*

In this section we analyze the singular behavior of the solutions through the nature of Riemann curvature invariant R and Kretschmann invariant S.

5.1. The Riemann curvature invariant for the GPR (with $n=0$) vacuum solution (2.18) is given by

$$R=-\frac{m_1^2 m_2 r^4}{(ar^2+b)^4}[8ab+\tfrac{1}{2}(p^2+3K^2)]\exp\left\{\frac{K+p}{(ab)^{1/2}}\tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \tag{5.1}$$

Thus we see that

$$R\to 0 \quad \text{as} \quad r\to 0 \quad \text{or} \quad r\to\infty \tag{5.2}$$

The Kretschmann curvature invariant S for the GPR (with $n=0$) vacuum solution (2.18) is given by

$$S=\frac{m_1^4 m_2^2 r^8}{(ar^2+b)^4}\left\{2\left[\frac{4a-2arA}{ar^2+b}+\frac{A}{r^2}+\frac{4}{r^2}-\frac{2}{r}\right]^2\right.$$
$$+\frac{B^2}{r}\left[A-B-\frac{4}{r}+\frac{4ar}{ar^2+b}\right]^2+\frac{B^2}{2}\left(A-\frac{2}{r}\right)^2$$
$$\left.+\frac{A^2}{4}\left(A-\frac{4}{r}\right)^2\right\}\exp\left\{\frac{2(K+p)}{(ab)^{1/2}}\tan^{-1}\left[\left(\frac{a}{b}\right)^{1/2} r\right]\right\} \tag{5.3}$$

where

$$A = \frac{4ar - (K+p)}{ar^2 + b}$$

$$B = \frac{p-K}{ar^2+b}$$

Now, it is clear that

$$\begin{aligned} S &\to \infty \quad \text{when} \quad r \to 0 \\ S &\to 0 \quad \text{when} \quad r \to \infty \end{aligned} \tag{5.4}$$

5.2. The Riemann curvature invariant R for the solution (3.21) is

$$R = 0 \tag{5.5}$$

The Kretschmann curvature invariant S for the solution (3.21) is

$$S = \frac{4F^2}{(F-1)^2 r^4} \left\{ \frac{2(1-F)(F+2F+1)}{(F+F)^2} + \frac{6}{F} - \frac{1}{F^2} + \left[3 - \frac{(F-1)(3F+F)}{F(F+F)} \right]^2 \right\} \tag{5.6}$$

where

$$F = K^2 r^2 + 1$$

It is obvious that

$$\begin{aligned} S &\to \infty \quad \text{when} \quad r \to 0 \\ S &\to 0 \quad \text{when} \quad r \to \infty \end{aligned} \tag{5.7}$$

6. CONCLUSIONS

Some static spherically symmetric solutions have been obtained for the GPR for both $n = 0$ and $n \neq 0$. These solutions have a naked singularity at $r = 0$ in one case for $n < 1$ and in the other case for arbitrary values of n. Thus, at least within the subset of vacuum spacetimes of spherical symmetry, there are no black holes in Arcidiacono's (1986, 1987) general projective relativity.

APPENDIX A

The Riemann curvature tensors R_{hijk} for the metric (2.1) are

$$R_{1212}=\tfrac{1}{2}r^2\, e^{\beta}\left(\beta_{11}+\frac{\beta_1}{r}\right)$$

$$R_{1313}=\tfrac{1}{2}r^2 \sin^2\theta\, e^{\beta}\left(\beta_{11}+\frac{\beta_1}{r}\right)$$

$$R_{1414}=\frac{e^{\alpha}}{2}[-\alpha_{11}+\tfrac{1}{2}\alpha_1(\beta_1-\alpha_1)]$$

$$R_{2323}=\tfrac{1}{4}r^4 \sin^2\theta\, e^{\beta}\left(\beta_1^2+\frac{4\beta_1}{r}\right)$$

$$R_{2424}=-\tfrac{1}{4}r^2\, e^{\alpha}\left(\alpha_1\beta_1+\frac{2\alpha_1}{r}\right)$$

$$R_{3434}=-\tfrac{1}{4}r^2 \sin^2\theta\left(\alpha_1\beta_1+\frac{2\alpha_1}{r}\right)$$

The Riemann and Kretschmann curvature invariants R and S for this metric are

$$R=-e^{-\beta}\left[2\beta_{11}+\alpha_{11}+\tfrac{1}{2}(\alpha_1^2+\beta_1^2+\alpha_1\beta_1)+\frac{2}{r}(2\beta_1+\alpha_1)\right]$$

$$S=e^{-2\beta}\left\{2\left(\beta_{11}+\frac{\beta_1}{2}\right)^2+\tfrac{1}{4}[\alpha_1(\beta_1-\alpha_1)-2\alpha_{11}]^2\right.$$
$$\left.+\frac{1}{2}\left(\beta_1+\frac{2}{r}\right)^2\alpha_1^2+\frac{1}{4}\left(\beta_1^2+\frac{4\beta_1}{r}\right)^2\right\}$$

respectively.

APPENDIX B

The Riemann curvature tensors R_{hijk} for the metric (3.1) are

$$R_{1212}=-\tfrac{1}{2}r\beta_1$$
$$R_{1313}=-\tfrac{1}{2}r\sin^2\theta\,\beta_1$$
$$R_{1414}=-\tfrac{1}{2}e^{\alpha}[\alpha_{11}+\tfrac{1}{2}\alpha_1(\alpha_1-\beta_1)]$$
$$R_{2323}=r^2\sin^2\theta(e^{-\beta}-1)$$
$$R_{2424}=-\tfrac{1}{2}r\alpha_1\, e^{\alpha-\beta}$$
$$R_{3434}=-\tfrac{1}{2}r\sin^2\theta\,\alpha_1\, e^{\alpha-\beta}$$

The Riemann curvature invariant R is

$$R = e^{-\beta}\left[\alpha_{11} + \tfrac{1}{2}\alpha_1(\alpha_1 - \beta_1) + \frac{2}{r}\left(\alpha_1 - \beta_1 + \frac{1}{r} - \frac{e^{\beta}}{r}\right)\right]$$

and the Kretschmann curvature invariant S is

$$S = 4\, e^{-2\beta}\left[\frac{(1-e^{\beta})^2}{r^4} + \frac{1}{2r^2}(\alpha_1^2 + \beta_1^2) + (\tfrac{1}{2}\alpha_{11} + \tfrac{1}{4}\alpha_1^2 - \tfrac{1}{4}\alpha_1\beta_1)^2\right]$$

ACKNOWLEDGMENTS

The authors wish to place on record their sincere thanks to the CSIR, New Delhi, for financial support of this work through Research Project No. 25(31)/87-EMR-II.

REFERENCES

Arcidiacono, G. (1986). *Relativity, Cosmology and Gravitation*, Hadronic Press, Massachusetts.
Arcidiacono, G. (1987). *Hadronic Journal*, **10**, 87
Brans, C., and Dicke, R. H. (1961). *Physical Review*, **124**, 925.
Singh, T., and Arcidiacono, G. (1989). A Birkhoff-type theorem in general projective relativity, preprint.
Singh, T., and Singh, G. P. (1989). *Astrophysics and Space Science*, **159**, 85.
Singh, T., Singh, G. P., and Arcidiacono, G. (1989). *Hadronic Journal*, **28**, 835.
Synge, J. L. (1960). *Relativity: The General Theory*, North-Holland, Amsterdam.
Vanden Bergh, N. (1980). *General Relativity and Gravitation*, **12**, 863.